

On-shell *versus* Off-shell Equivalence in 3D Gravity

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ABSTRACT

A given field theory action determines a set of field equations but other actions may yield equivalent field equations; if so they are *on-shell* equivalent. They may also be *off-shell* equivalent, being related by the elimination of auxiliary fields or by local field redefinitions, but this is not guaranteed, as we demonstrate by consideration of the linearized limit of 3D massive gravity models. Failure to appreciate this subtlety has led to incorrect conclusions in recent studies of the “Minimal Massive Gravity” model.

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1 Introduction

A classical field theory is fully specified by a set of field equations but it is usually possible to find an action from which the field equations may be derived as the conditions for stationarity. Obviously, these conditions are unchanged by a change of either the sign or the scale of the action but the sign is usually fixed by the requirement of positive energy; or unitarity of the semi-classical theory, and in this context the scale corresponds to choice of units for \hbar . As a result, there is usually no ambiguity in the choice of action for an *interacting* semi-classical field theory if actions related by field redefinitions and/or elimination of auxiliary fields are considered equivalent. Ambiguities in *relative* scales and signs may arise after linearization but these can usually be resolved by reference to the interactions, or by symmetries inherited from the interacting theory. In the context of gauge theories, or gravity, the requirements of gauge invariance and semi-classical unitarity are usually sufficient to eliminate ambiguities.

For these reasons, little attention has been paid to ambiguities arising in the choice of action for fields subject to a given set of field equations. However, this issue has become relevant recently in the context of 3D massive gravity models; in particular Topologically Massive Gravity (TMG) [1] and Minimal Massive Gravity (MMG) [2]. This is partly because the field equations of TMG and MMG are third order, rather than second order, and partly because of the “third-way consistency” of the MMG equations (as reviewed in [3]).

A characteristic of gauge/gravity field equations that are third-way consistent is that their off-shell extension requires an action with auxiliary fields that cannot be consistently eliminated from the action, even though (by definition of “auxiliary”) they can be eliminated from the field equations. The MMG model of [2], which is

a simple modification of TMG, was the first example. This was followed by a gauge theory example in which the usual 3D $SU(2)$ Yang-Mills equation is modified in a similar way [4]; the action is then equivalent to the difference of two $SU(2)$ Chern-Simons actions plus a mass term that breaks $SU(2) \times SU(2)$ to the diagonal $SU(2)$ subgroup, which is an action originally proposed in the context of multiple M2/D2-brane dynamics [5].

A key issue for this paper is what happens to third-way consistent theories upon linearization, and the spin-1 example of [4] provides a convenient starting point. Linearization of the field equations yields a triplet of Maxwell actions, as for the standard $SU(2)$ -YM theory, and this suggests that the two quadratic actions must be equivalent. This is indeed the case, as is shown by a simple local field redefinition. In this spin-1 case, no distinction arises between on-shell equivalence and off-shell equivalence.

Turning to the spin-2 case, we reconsider the linearization of MMG about an AdS_3 vacuum. The field equations become equivalent to those of linearized TMG in this vacuum, for a rescaled mass, and the corresponding quadratic actions are therefore *on-shell* equivalent. If one assumes that there is no *off-shell* ambiguity then the quadratic MMG action must be the quadratic TMG action with the rescaled mass; this leads to the conclusion of [6] (recently reiterated in [7]) that the unitarity problems of TMG with AdS asymptotics are shared by MMG, thus contradicting the main result of [2]. The problem with this conclusion is that the premise of no off-shell ambiguity is, in this case, false.

The quadratic action for MMG in its AdS_3 vacuum was obtained in [2] in a form that leads directly to first-order equations. In this form, the Virasoro central charges of the asymptotic conformal symmetry algebra can be read off from the coefficients of the terms that survive in the limit of infinite graviton mass (in which limit the result of Brown and Henneaux [8] for 3D GR is recovered). This step was dealt with briefly in [2] and one purpose of this paper is to give a detailed derivation. We also explain precisely how this quadratic action is related to an action that directly yields the third-order linearized MMG equations.

However, it is another equivalent form of the quadratic action for linearized MMG that is most useful to a discussion of the issue of on-shell *versus* off-shell equivalence. This action is found by a local field redefinition (defined away from the chiral point) followed by an elimination of variables; it is the sum of a linearized Einstein-Hilbert term and a standard first-order action for a single free spin-2 mode in AdS_3 , with coefficients that depend on the MMG parameters. This action makes clear how off-shell ambiguities can arise because the *relative sign* between the two terms in this quadratic action cannot be changed by any local field redefinition. Any two distinct MMG models will be off-shell *inequivalent* if they differ in this relative sign.

The physical parameter space of semi-classical MMG is two-dimensional, and MMG degenerates to TMG on a one-parameter curve in this space. On this ‘TMG curve’ the relative sign in the quadratic action is fixed, and this is what leads to the unitarity problem of TMG in an AdS_3 vacuum: the so-called “bulk/boundary clash”. However,

there is a region in the MMG parameter space in which the relative sign is opposite to that of TMG, and this is how MMG evades the “bulk/boundary clash”. Within a subregion (which is connected once account is taken of an equivalence relation in parameter space [9]) all semi-classical unitarity conditions are satisfied [2].

TMG/MMG is not the only pair of 3D gravity theories that become on-shell equivalent in the linearized limit but remain off-shell inequivalent, nor is it necessary for the background to be AdS. In our concluding section we discuss another pair of massive 3D gravity theories that become on-shell equivalent when linearized about their Minkowski vacua but for which off-shell equivalence, even in this linearized limit, is *a priori* obvious!

2 The spin-1 case

The gauge-potential one-form A of an $SU(2)$ Yang-Mills (YM) theory is a 3-vector in the Lie algebra of $SU(2)$. Its two-form field strength is

$$dA + \frac{1}{2}A \times A = F \equiv \frac{1}{2}dx^\mu dx^\nu F_{\mu\nu}, \quad (2.1)$$

where we use the cross product notation of 3-vector algebra. We shall suppose that A has dimensions of mass in units for which $\hbar = 1$ so that F has dimension of mass-squared. For a 3D background Minkowski spacetime, with metric η and standard Minkowski coordinates, the first-order form of the standard 3D YM Lagrangian density is

$$\mathcal{L}_{\text{YM}} = G_\mu \cdot \tilde{F}^\mu - \frac{1}{2}\eta^{\mu\nu}G_\mu \cdot G_\nu \quad \left(\tilde{F}^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho}F_{\nu\rho} \right), \quad (2.2)$$

where G_μ is an auxiliary $SU(2)$ triplet of Lorentz vectors (of dimension mass-squared) and we use the dot product notation of 3-vector algebra to construct an $SU(2)$ singlet. Elimination of G from this action, by means of its field equation $G = \tilde{F}$, yields the standard second-order YM Lagrangian density in terms of \tilde{F} , which leads to the standard 3D YM field equation in the form

$$\varepsilon^{\mu\nu\rho}D_\nu\tilde{F}_\rho = 0, \quad (2.3)$$

where D is the covariant derivative with gauge potential A . This equation implies, as a consequence of the Lie algebra Jacobi identity and the symmetry of mixed partial derivatives, that

$$D_\mu \left[\varepsilon^{\mu\nu\rho}D_\nu\tilde{F}_\rho \right] \equiv 0, \quad (2.4)$$

which is the Noether identity required by gauge invariance of the YM action.

Consider now the following modified YM field equation [4]:

$$\varepsilon^{\mu\nu\rho}D_\nu\tilde{F}_\rho + \frac{1}{2m}\varepsilon^{\mu\nu\rho}\tilde{F}_\nu \times \tilde{F}_\rho = 0, \quad (2.5)$$

where m is a mass parameter. In light of the identity (2.4), consistency requires that

$$D_\mu \left[\varepsilon^{\mu\nu\rho} \tilde{F}_\nu \times \tilde{F}_\rho \right] = 0, \quad (2.6)$$

but this is *not* an identity. If it were an identity then we might expect to be able to find an action $I[A]$ from which the modified YM field equation could be derived by variation¹; as it is not an identity, no such action exists! Nevertheless, there *is* an action involving the auxiliary 3-vector G ; its Lagrangian density is [4]

$$\mathcal{L} = \mathcal{L}_{\text{YM}} + \frac{1}{2m} \varepsilon^{\mu\nu\rho} \left(G_\mu \cdot D_\nu G_\rho + \frac{1}{3m} G_\mu \cdot G_\nu \times G_\rho \right). \quad (2.7)$$

It appears that G is no longer auxiliary, but the field equations of A and G are jointly equivalent to $G = \tilde{F}$ and (2.5). The attempt to find an action $I[A]$ by setting $G = \tilde{F}$ in (2.7) fails because this equation for G is a linear combination of the field equations found from variation of G and A , not the one found from variation of G alone. This is characteristic of gauge field equations that are “third-way consistent”.

One might wonder whether the modification to the YM Lagrangian density in (2.7) could be cancelled by a field redefinition of the form $A = A' - (\alpha/m)G$ for some constant α . Clearly, one may remove *either* the $G \cdot DG$ term *or* the $G \cdot G \times G$ term in this way, but one cannot remove both. For example, by choosing $\alpha = 1/2$ we arrive at the new Lagrangian density

$$\mathcal{L} = \mathcal{L}_{\text{YM}} - \frac{1}{12m^2} \varepsilon^{\mu\nu\rho} G_\mu \cdot G_\nu \times G_\rho. \quad (2.8)$$

The field equation for G is now algebraic, but still non-linear; it can be solved by an infinite series in powers of \tilde{F}/m^2 [5] but there is no guarantee of convergence.

The above discussion is easily generalized to other gauge groups \mathcal{G} , and the choice $\mathcal{G} = U(1)$ is directly relevant to linearization of the $\mathcal{G} = SU(2)$ choice, we shall now focus on this. It is obvious that the modified YM field equation of (2.5) degenerates to the 3D Maxwell equation for $\mathcal{G} = U(1)$ but this still leaves open the possibility of inequivalent actions, and one might expect to find a new non-standard action from the $\mathcal{G} = U(1)$ variant of (2.7). However, the alternative Lagrangian density (2.8) makes it clear that this does not happen because the cubic term in G is absent for $\mathcal{G} = U(1)$; the candidate for a new 3D Maxwell action is related to the standard action by a local field redefinition. In this linearized spin-1 case, on-shell equivalence implies off-shell equivalence.

3 Spin-2 in AdS₃

A complication of the spin-2 case is that the vacuum spacetime must now be found as a solution of the field equations. Here we shall be concerned with massive gravity models

¹Although there is no theorem that guarantees this [10].

that admit an AdS_3 vacuum. These include TMG and MMG, and other massive gravity theories that we shall discuss in our concluding section. We shall see that linearization of TMG and MMG about an AdS_3 vacuum leads to massive spin-2 field equations that are equivalent except for the dependence of the mass parameter on the parameters that define the TMG and MMG theories. This is to be expected from the fact that MMG involves an additional parameter. However, we shall also see that the higher-dimensional parameter space of MMG leads to an off-shell inequivalence within this parameter space, which allows MMG to be unitary, at least at the semi-classical level, even though TMG is not.

3.1 TMG and MMG actions

In first-order (Chern-Simons-like) formulation, the TMG Lagrangian can be written in terms of three Lorentz-vector valued one-form fields: the dreibein e^a , a dualized spin-connection ω^a and an auxiliary field h^a , for $a = 0, 1, 2$. Using the dot and cross notation for contractions with the invariant bilinear form (η_{ab}) and structure constants (ϵ_{abc}) of $so(2, 1)$, following [2], we may write the Riemann curvature two-form and torsion two-form as, respectively,

$$R(\omega) = d\omega + \frac{1}{2}\omega \times \omega, \quad T(\omega) = de + \omega \times e, \quad (3.1)$$

and the TMG Lagrangian 3-form as

$$L_{\text{TMG}} = -\sigma e \cdot R(\omega) + \frac{\Lambda_0}{6} e \cdot e \times e + h \cdot T(\omega) + \frac{1}{\mu} L_{\text{LCS}}(\omega), \quad (3.2)$$

where σ is a sign², Λ_0 is a ‘cosmological parameter’, μ a mass parameter, and L_{LCS} is the Lorentz-Chern-Simons 3-form:

$$L_{\text{LCS}}(\omega) = \frac{1}{2} \omega \cdot \left(d\omega + \frac{1}{3} \omega \times \omega \right). \quad (3.3)$$

As this term breaks parity, we may choose $\mu > 0$ without loss of generality.

After elimination of the auxiliary field h and the dualized spin connection ω by their field equations, the TMG action can be written in terms of the metric alone. In this form it is the sum of an Einstein-Hilbert term and a Lorentz-Chern-Simons term [1]; the first of these has a coefficient inversely proportional to the 3D Newton constant G_3 , which has dimensions of inverse mass. The cosmological parameter Λ_0 is, for TMG, the cosmological constant Λ in a maximally symmetric background. Here we shall be interested in the AdS_3 case, for which $\Lambda = -1/\ell^2$, where ℓ is the AdS radius of curvature. The semi-classical limit is one for which

$$\frac{\ell}{\hbar G_3} \rightarrow \infty, \quad \mu G_3 \rightarrow 0, \quad \text{for fixed } \frac{\mu \ell}{\hbar}. \quad (3.4)$$

²One may allow σ to be any real number, and this simplifies the description of the MMG parameter space [9], but here we follow [2].

The parameter space of semi-classical TMG is therefore one-dimensional, and it is parametrized by the dimensionless parameter $\mu\ell/\hbar$ [11].

MMG is defined by the following simple modification of the TMG Lagrangian 3-form:

$$L_{\text{MMG}} = L_{\text{TMG}} + \frac{\alpha}{2} e \cdot h \times h, \quad (3.5)$$

where α is a new dimensionless parameter. The parameter-space of semi-classical MMG is therefore two-dimensional and MMG degenerates to TMG on the one-dimensional subspace defined by $\alpha = 0$. For $\alpha \neq 0$ it is still true that h and ω can be eliminated from the field equations, which can therefore be written in terms of the metric alone, but the auxiliary field h can no longer be eliminated from the *action*. To understand this unusual state of affairs it is convenient to express the action in terms of the new (dualized) connection

$$\Omega = \omega + \alpha h. \quad (3.6)$$

The action is slightly more complicated in terms of the connection Ω ; it reads³

$$\begin{aligned} L_{\text{MMG}} = & -\sigma e \cdot R(\Omega) + \frac{\Lambda_0}{6} e \cdot e \times e + (1 + \alpha\sigma) h \cdot T(\Omega) - \frac{\alpha}{2} (1 + \alpha\sigma) e \cdot h \times h \\ & + \frac{1}{\mu} L_{\text{LCS}}(\Omega) - \frac{\alpha}{\mu} h \cdot \left(R(\Omega) - \frac{\alpha}{2} D(\Omega) h + \frac{\alpha^2}{6} h \times h \right). \end{aligned} \quad (3.7)$$

Here $D(\Omega)$ denotes the covariant derivative with respect to the connection Ω . The field equations are:

$$-\sigma R(\Omega) + (1 + \alpha\sigma) D(\Omega) h - \frac{\alpha}{2} (1 + \alpha\sigma) h \times h + \frac{\Lambda_0}{2} e \times e = 0, \quad (3.8a)$$

$$-\sigma T(\Omega) + \frac{1}{\mu} R(\Omega) - \frac{\alpha}{\mu} D(\Omega) h + (1 + \alpha\sigma) e \times h + \frac{\alpha^2}{2\mu} h \times h = 0, \quad (3.8b)$$

$$(1 + \alpha\sigma) T(\Omega) - \frac{\alpha}{\mu} R(\Omega) + \frac{\alpha^2}{\mu} D(\Omega) h - \alpha(1 + \alpha\sigma) e \times h - \frac{\alpha^3}{2\mu} h \times h = 0. \quad (3.8c)$$

By taking linear combinations of these equations, one can show that they are equivalent to the following simpler set:

$$T(\Omega) = 0, \quad (3.9a)$$

$$R(\Omega) + \mu(1 + \alpha\sigma)^2 e \times h + \frac{\Lambda_0 \alpha}{2} e \times e = 0, \quad (3.9b)$$

$$D(\Omega) h + \sigma\mu(1 + \alpha\sigma) e \times h - \frac{\alpha}{2} h \times h + \frac{\Lambda_0}{2} e \times e = 0. \quad (3.9c)$$

The first of these equations tells us that the connection Ω is torsionless; it can be solved for in terms of de . The second equation allows us to solve for h in terms of $R(\Omega)$ and e . Substituting these solutions into the third equation leads to the MMG field equation presented in [2]:

$$\frac{1}{\mu} C_{\mu\nu} + \bar{\sigma} G_{\mu\nu} + \bar{\Lambda}_0 g_{\mu\nu} = -\frac{\gamma}{\mu^2} J_{\mu\nu}, \quad (3.10)$$

³We assume, as in [2], that $(1 + \alpha\sigma) \neq 0$.

where $C_{\mu\nu}$ is the Cotton tensor, $G_{\mu\nu}$ the Einstein tensor and $J_{\mu\nu}$ a curvature squared symmetric tensor given by

$$J_{\mu\nu} = \frac{1}{2 \det g} \varepsilon_\mu^{\rho\sigma} \varepsilon_\nu^{\tau\eta} S_{\rho\tau} S_{\sigma\eta}, \quad S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R. \quad (3.11)$$

The coefficients appearing in the MMG field equations (3.10) are related to the coefficients of the Lagrangian 3-form (3.5) by

$$\begin{aligned} \bar{\sigma} &= \sigma + \alpha \left[1 + \frac{\alpha \Lambda_0 / \mu^2}{2(1 + \sigma\alpha)^2} \right], & \gamma &= -\frac{\alpha}{(1 + \sigma\alpha)^2}, \\ \bar{\Lambda}_0 &= \Lambda_0 \left[1 + \sigma\alpha - \frac{\alpha^3 \Lambda_0 / \mu^2}{4(1 + \sigma\alpha)^2} \right]. \end{aligned} \quad (3.12)$$

These manipulations are fine at the level of field equations, but they cannot be used to obtain an action $I[g]$ for which variation with respect to the metric g yields the MMG metric equation (3.10). The reason for this is that a linear combination of all field equations had to be used when solving for h and Ω , so back-substitution in the action is not legitimate; it leads to an *inequivalent* action and a corresponding field equation that is inequivalent to (3.10).

3.2 AdS vacuum and Linearization

For an AdS_3 vacuum solution of the MMG field equations (3.9), we have

$$R(\Omega) = \frac{\Lambda}{2} e \times e, \quad h = C \mu e, \quad (3.13)$$

where Λ is the cosmological constant and C a dimensionless constant. These constants are related to each other and the parameters of the action by

$$C = -\frac{(\Lambda + \alpha \Lambda_0) / \mu^2}{2(1 + \alpha\sigma)^2} \quad (3.14)$$

and

$$(\Lambda_0 - \sigma\Lambda) / \mu^2 - \alpha(1 + \sigma\alpha)C^2 = 0. \quad (3.15)$$

Let $e = \bar{e}$ be a given AdS_3 vacuum solution⁴ and $\Omega = \bar{\Omega}$ the corresponding dual spin-connection 1-form. We expand about this background by setting

$$e = \bar{e} + k, \quad \Omega = \bar{\Omega} + v, \quad h = C\mu(\bar{e} + k) + p. \quad (3.16)$$

The linearized field equations may now be found by expanding the full field equations (3.9) to first order in the perturbation one-forms (k, v, p) . We may also arrive at these

⁴There may be none, one or two, depending on the choice of parameters.

linearized equations by first expanding the Lagrangian 3-form of (3.7) to second order; using (3.14) and (3.15), we find that

$$\begin{aligned}
L_{MMG}^{(2)} = & -(\sigma + \alpha C) \left[k \cdot \bar{D}v + \frac{1}{2} \bar{e} \cdot v \times v - \frac{\Lambda}{2} \bar{e} \cdot k \times k \right] \\
& - \frac{\Lambda}{\mu} \left[\bar{e} \cdot v \times k + \frac{1}{2} k \cdot \bar{D}k - \alpha \bar{e} \cdot k \times p \right] \\
& + (1 + \alpha\sigma + \alpha^2 C) \left[p \cdot \bar{D}k + \bar{e} \cdot v \times p - \frac{\alpha}{2} \bar{e} \cdot p \times p \right] \\
& + \frac{1}{\mu} \left[\left(\frac{1}{2} v - \alpha p \right) \cdot \bar{D}v + \frac{\alpha^2}{2} p \cdot \bar{D}p \right],
\end{aligned} \tag{3.17}$$

where \bar{D} is the covariant derivative with respect to the background spin-connection $\bar{\Omega}$.

The linearized MMG field equations now follow by variation of the quadratic MMG action with respect to (k, v, p) ; the resulting equations are jointly equivalent to

$$\bar{D}k + \bar{e} \times v = 0, \tag{3.18a}$$

$$\bar{D}v - \Lambda \bar{e} \times k = -\mu(1 + \alpha\sigma)^2 \bar{e} \times p, \tag{3.18b}$$

$$\bar{D}p + M \bar{e} \times p = 0, \tag{3.18c}$$

where M (the mass of the spin-2 mode) is given by

$$M = \mu(\sigma(1 + \alpha\sigma) - \alpha C). \tag{3.19}$$

These equations should be equivalent to those found by linearization of the third-order metric field equation (3.10). To verify this, we first observe that equations (3.18) imply the constraints

$$\bar{e} \cdot k = \bar{e} \cdot v = \bar{e} \cdot p = 0. \tag{3.20}$$

If we define

$$k_{\mu\nu} \equiv k_\mu{}^a \bar{e}_\nu{}^b \eta_{ab}, \tag{3.21}$$

and likewise for the other fields, then the constraints state that the two-tensor fields (k, v, p) are symmetric. We may solve equations (3.18a) and (3.18b) for the symmetric two-tensors v and p :

$$v_{\mu\nu} = \det(\bar{e})^{-1} \epsilon_\nu{}^{\alpha\beta} \bar{\nabla}_\alpha k_{\beta\mu}, \quad p_{\mu\nu} = \frac{2}{\mu(1 + \alpha\sigma)^2} \left(\mathcal{G}_{\mu\nu}(k) - \frac{1}{2} \bar{g}_{\mu\nu} \mathcal{G}_\lambda{}^\lambda(k) \right), \tag{3.22}$$

where $\mathcal{G}_{\mu\nu}(k)$ is the linearized Einstein tensor and $\bar{g}_{\mu\nu} = \bar{e}_\mu{}^a \bar{e}_\nu{}^b \eta_{ab}$ is the background AdS₃ metric. Equation (3.18c) then becomes

$$\epsilon_\mu{}^{\alpha\beta} \bar{\nabla}_\alpha \mathcal{G}_{\beta\nu}(k) + M \mathcal{G}_{\mu\nu}(k) = 0. \tag{3.23}$$

This is indeed equivalent to the equation that one obtains from direct linearization of (3.10). It is also equivalent to the linearized TMG equation, albeit with a different value for the mass of the spin-2 mode. It is therefore tempting to suppose that the

quadratic action of linearized MMG must be equivalent to the quadratic action of linearized TMG. This was the premise of [6], which led to the conclusion that the known unitarity problems of TMG persist in MMG. We show here that this reasoning is mistaken because the quadratic action of linearized MMG is *inequivalent* to the quadratic action of TMG.

3.3 MMG *versus* TMG

For any massive 3D gravity model with AdS asymptotics, semi-classical unitarity requires positive energy of the bulk spin-2 modes *and* positive Virasoro central charges for the asymptotic conformal algebra. For TMG it is not possible to satisfy both these conditions simultaneously. The standard “wrong sign” choice for the Einstein Hilbert term in the standard TMG action ensures that the spin-2 mode is physical but this comes at the cost of positivity of the central charges; at least one must be negative. Changing the overall sign of the action (thereby restoring the “right sign” for the Einstein-Hilbert term) will allow both central charges to be positive but this now comes at the cost of negative energy for the bulk spin-2 mode. The main result of [2] is that this “bulk/boundary clash” is resolved by MMG; we shall re-investigate this claim in a way that clarifies its relation to the issue of on-shell versus off-shell equivalence.

In what follows, we shall assume that the overall sign of the action has been chosen such that the Virasoro central charges are positive. We introduce a new set of one-form fields $(\tilde{k}, \tilde{v}, p)$ to replace the one-form fields (k, v, p) of (3.17) by setting

$$k = \frac{1}{2}(\sqrt{\lambda_-} - \sqrt{-\lambda_+})\tilde{k} + \frac{\ell}{2}(\sqrt{\lambda_-} + \sqrt{-\lambda_+})\tilde{v} + \frac{1}{\mu(1-2C)}p \quad (3.24a)$$

$$v = \frac{1}{2\ell}(\sqrt{\lambda_-} + \sqrt{-\lambda_+})\tilde{k} + \frac{1}{2}(\sqrt{\lambda_-} - \sqrt{-\lambda_+})\tilde{v} + \frac{M}{\mu(1-2C)}p \quad (3.24b)$$

where

$$\lambda_{\pm} = 1 \mp (\sigma + \alpha C)\mu\ell. \quad (3.25)$$

This is an invertible field redefinition provided that

1. $\mp\lambda_{\pm} > 0$. As we explain in the following section, this is equivalent to positivity of both Virasoro central charges. We note here that this implies that

$$\lambda_+\lambda_- < 0. \quad (3.26)$$

2. $1 - 2C \neq 0$. From the identity

$$1 - 2C \equiv \frac{(M\ell)^2 - 1}{(1 + \sigma\alpha)^2(\mu\ell)^2}, \quad (3.27)$$

we see that this condition is equivalent to $|M\ell| \neq 1$. In other words, the change of variables is defined away from the “chiral point” $|M\ell| = 1$.

In terms of the new set of one-form fields $(\tilde{k}, \tilde{v}, p)$, the Lagrangian 3-form of (3.17) takes the form

$$L_{MMG}^{(2)} = \frac{\lambda_+ \lambda_-}{\mu \ell^2} \left(\tilde{k} \cdot \bar{D} \tilde{v} + \frac{1}{2} \bar{e} \cdot \tilde{v} \times \tilde{v} + \frac{1}{2 \ell^2} \bar{e} \cdot \tilde{k} \times \tilde{k} \right) + \frac{1}{2\mu(1-2C)} [p \cdot \bar{D} p + M \bar{e} \cdot p \times p] . \quad (3.28)$$

Varying \tilde{v} now yields the equation $\bar{D} \tilde{k} + \bar{e} \times \tilde{v} = 0$, which can be solved for \tilde{v} ; the corresponding two-tensor is symmetric and given by (3.22) with k and v replaced by \tilde{k} and \tilde{v} . Using this solution for \tilde{v} we arrive at the following equivalent quadratic action for linearized MMG:

$$L_{MMG}^{(2)} = \frac{\lambda_+ \lambda_-}{\mu \ell^2} \tilde{k}^{\mu\nu} \mathcal{G}_{\mu\nu}(\tilde{k}) + \frac{1}{2\mu(1-2C)} [p^{\mu\nu} \epsilon_\mu^{\alpha\beta} \bar{\nabla}_\alpha p_{\beta\nu} + M(p^{\mu\nu} p_{\mu\nu} - p^2)] . \quad (3.29)$$

This action is the sum of two terms: a second-order action for linearized 3D gravity with metric perturbation \tilde{k} , which contains no gauge-invariant local degrees of freedom, and a first-order action for p that describes a single spin-2 mode of mass M . Further field redefinitions can change the magnitudes of the coefficients of these terms but not their signs. Our initial assumption of positive Virasoro central charges has fixed the sign of the coefficient of the \tilde{k} term, because it implies that $\lambda_+ \lambda_- < 0$, but either sign remains possible for the coefficient of the other term, and this leads to the possibility of off-shell inequivalence.

For MMG it is possible to choose parameters such that the sign of the first-order action for p is either the same as or opposite to the sign for the \tilde{k} term. The bulk/boundary clash can be resolved only if the signs are the same; this condition just restricts the parameter space of MMG but it cannot be satisfied by TMG because TMG is the $\alpha \rightarrow 0$ limit of MMG and

$$\mu(1-2C) \stackrel{\alpha \rightarrow 0}{\rightarrow} \mu \left(\sigma - \frac{1}{\mu \ell} \right) \left(\sigma + \frac{1}{\mu \ell} \right) = - \frac{\lambda_+ \lambda_-}{\mu \ell^2} \Big|_{\alpha=0} . \quad (3.30)$$

What we wish to stress here is that the quadratic action for linearized TMG, with its opposite signs for the two independent terms in the quadratic Lagrangian 3-form (3.29), is *inequivalent* to the quadratic action for linearized MMG when its parameters are chosen such the signs are the same, as required by unitarity.

We have stated that the Virasoro central charges in the asymptotic conformal symmetry algebra are both positive when $\mp \lambda_\pm > 0$. Although this fact did not play an essential role in the above analysis, it is necessary to know how the parameters λ_\pm are related to the central charges if one wishes to read them off from the quadratic action. This relation was explained briefly in [2]; in the following section we provide a more complete derivation.

4 Asymptotic symmetries in CS-like theories

The action for MMG belongs to the class of theories with a Chern-Simons-like formulation [12, 13]. These models can be defined in terms of $so(2, 1)$ -vector valued one-form fields with a bulk action resembling a Chern-Simons theory; these are now included as special cases. For a review and Hamiltonian analysis of this type of theory we refer to [13, 14]. Here we first recall some results presented in [15], where the procedure of computing the asymptotic symmetry algebra in general Chern-Simons-like theories was presented. and we then use this to rederive the central charges in MMG for asymptotically AdS_3 boundary conditions and compare with the results of [2] and [6]. We also explain how these results determine thermodynamic properties of the BTZ black hole in the context of a given CS-like theory, in particular MMG.

4.1 The algebra of asymptotic charges

CS-like models can be defined in terms of a set of $so(2, 1)$ -vector valued one-form fields labeled by field-space indices $\mathbf{p}, \mathbf{q}, \mathbf{r}$, with an action reminiscent of a CS theory:

$$I = \frac{k}{4\pi} \int \left(g_{\mathbf{pq}} a^{\mathbf{p}} \cdot da^{\mathbf{q}} + \frac{1}{3} f_{\mathbf{pqr}} a^{\mathbf{p}} \cdot a^{\mathbf{q}} \times a^{\mathbf{r}} \right). \quad (4.1)$$

Here $g_{\mathbf{pq}}$ and $f_{\mathbf{pqr}}$ are a completely symmetric field-space metric and structure constants, respectively and k is the overall coupling constant of the theory. As in section 3, we are suppressing wedge products and using dot and cross notation for the contraction of Lorentz indices with η_{ab} and ϵ_{abc} respectively.

The Chern-Simons-like action is invariant under diffeomorphisms by construction and it was shown in [15] that diffeomorphisms are generated by gauge-like transformations which take the fields $a^{\mathbf{p}} \rightarrow a^{\mathbf{p}} + \delta_{\xi} a^{\mathbf{p}}$ with

$$\delta_{\xi} a^{\mathbf{p}} = d\xi^{\mathbf{p}} + f^{\mathbf{p}}_{\mathbf{qr}} (a^{\mathbf{q}} \times \xi^{\mathbf{r}}). \quad (4.2)$$

When $\xi^{\mathbf{p}}$ is chosen as

$$\xi^{\mathbf{p}} = a_{\nu}^{\mathbf{p}} \zeta^{\nu}, \quad (4.3)$$

the transformation (4.2) generates diffeomorphisms on shell

$$\delta_{\zeta} a_{\mu}^{\mathbf{p}} = \zeta^{\nu} \partial_{\mu} a_{\nu}^{\mathbf{p}} + a_{\nu}^{\mathbf{p}} \partial_{\mu} \zeta^{\nu} + \dots \stackrel{\text{on-shell}}{=} \mathcal{L}_{\zeta} a_{\mu}^{\mathbf{p}}, \quad (4.4)$$

where the dots refer to terms which vanish by the equations of motion.

In the presence of boundaries, the constraint function generating bulk diffeomorphisms needs to be improved by a boundary term whose variation reads

$$\delta Q[\xi^{\mathbf{p}}] = \frac{k}{2\pi} \oint (g_{\mathbf{pq}} \xi^{\mathbf{p}} \cdot \delta a_{\varphi}^{\mathbf{q}}) d\varphi. \quad (4.5)$$

This defines the boundary charge of a diffeomorphism parameterized by (4.3).

In order to find the asymptotic symmetry algebra in the general CS-like theories we first specify the boundary conditions for our fields $a^{\mathbf{P}}$. They have to solve the field equations (at least asymptotically) and they should come equipped with the specification of what is allowed to fluctuate on the boundary and what is kept fixed; i.e., which components of the fields carry state-dependent information.

Then we determine the transformations (4.2) with gauge parameter (4.3) that preserve the boundary conditions, up to the transformation of state-dependent functions. In other words, on the left hand side of (4.2) we specify which components of the fields are allowed to fluctuate. Then we find the asymptotic gauge parameters $\xi^{\mathbf{P}}$ by solving for the right hand side of (4.2).

After having found the gauge parameters which preserve (4.2), the consistency of the boundary conditions can be checked by inserting the result for the gauge parameter into the variation of the charges (4.5). This should be finite on the boundary, integrable and conserved. Once these conditions are met, the Poisson brackets will solely receive contributions from the boundary charges on-shell and reduce to the Dirac bracket algebra of boundary charges [15]

$$\{Q[\xi^{\mathbf{P}}], Q[\eta^{\mathbf{Q}}]\}^* = -\delta_{\eta} Q[\xi^{\mathbf{P}}] = \frac{k}{\pi} \oint d\varphi \operatorname{tr} (g_{\mathbf{PQ}} \xi^{\mathbf{P}} \cdot \delta_{\eta} a_{\varphi}^{\mathbf{Q}}) \quad (4.6)$$

Imposing boundary conditions on $a_{\varphi}^{\mathbf{P}}$ suffices to determine the asymptotic symmetry algebra. The conditions on the radial component of the fields can be derived by solving the field equations asymptotically. The time components of the fields can then be found by demanding the boundary conditions on $a_{\varphi}^{\mathbf{Q}}$ to be conserved under time evolution.

4.2 Asymptotically AdS₃ boundary conditions in MMG

We will now investigate the asymptotic symmetry algebra for MMG when choosing asymptotically AdS₃ boundary conditions. These boundary conditions can be formulated by expanding the metric in Fefferman-Graham gauge, which in three dimensions leads to the Bañados metrics [16]

$$ds^2 = dr^2 - \ell^2 (e^{r/\ell} dx^+ - e^{-r/\ell} \mathcal{L}^-(x^-) dx^-) (e^{r/\ell} dx^- - e^{-r/\ell} \mathcal{L}^+(x^+) dx^+) , \quad (4.7)$$

where $x^{\pm} = t \pm \varphi$. We formulate our boundary conditions in terms of the dreibein in a suitable local Lorentz gauge. In terms of the generators T^a of the 3D Lorentz algebra $so(2, 1)$ we choose

$$e_{\varphi} = -\frac{\ell}{2} e^{-r/\ell} (\mathcal{L}^+ - \mathcal{L}^-) T^0 + \frac{\ell}{2} (2e^{r/\ell} + e^{-r/\ell} (\mathcal{L}^+ + \mathcal{L}^-)) T^1, \quad (4.8a)$$

$$e_t = \frac{\ell}{2} (2e^{r/\ell} - e^{-r/\ell} (\mathcal{L}^+ + \mathcal{L}^-)) T^0 + \frac{\ell}{2} e^{-r/\ell} (\mathcal{L}^+ - \mathcal{L}^-) T^1, \quad (4.8b)$$

$$e_r = T^2. \quad (4.8c)$$

The functions \mathcal{L}^{\pm} carry state dependent information and are allowed to fluctuate on the boundary. We wish to find the asymptotic symmetry algebra of diffeomorphisms

which preserve this form of the dreibein, up to $\mathcal{L}^\pm \rightarrow \mathcal{L}^\pm + \delta_\xi \mathcal{L}^\pm$. First, we need to solve the constraint equations

$$g_{\mathbf{pq}} da^{\mathbf{q}} + \frac{1}{2} f_{\mathbf{pqr}} a^{\mathbf{q}} \times a^{\mathbf{r}} = 0, \quad (4.9)$$

where $g_{\mathbf{pq}}$ and $f_{\mathbf{pqr}}$ are such that (4.1) gives the MMG action (3.5). The solution is given as

$$\omega = \Omega - \alpha h, \quad h = C\mu e, \quad (4.10)$$

with

$$\Omega_\varphi = \frac{1}{2} (-2e^{r/\ell} + e^{-r/\ell}(\mathcal{L}^+ + \mathcal{L}^-)) T^0 - \frac{1}{2} e^{-r/\ell}(\mathcal{L}^+ - \mathcal{L}^-) T^1, \quad (4.11a)$$

$$\Omega_t = \frac{1}{2} e^{-r/\ell}(\mathcal{L}^+ - \mathcal{L}^-) T^0 - \frac{1}{2} (2e^{r/\ell} + e^{-r/\ell}(\mathcal{L}^+ + \mathcal{L}^-)) T^1, \quad (4.11b)$$

$$\Omega_r = 0. \quad (4.11c)$$

We are now ready to compute the transformations (4.2) on the fields. We are assisted in this process by the secondary constraint of MMG, which reads

$$e \cdot h = 0. \quad (4.12)$$

This implies for gauge parameters $\xi^{\mathbf{p}} = a_\mu^{\mathbf{p}} \zeta^\mu$ that

$$e \cdot \xi^h = h \cdot \xi^e, \quad (4.13)$$

and hence, by (4.10), that $\xi^h = C\mu \xi^e$. The diffeomorphisms preserving the form of (4.8) are given by gauge parameters ξ^e and $\xi^\Omega = \xi^\Omega - \alpha C\mu \xi^e$ expressed in terms of two arbitrary functions $f^\pm(x^\pm)$

$$\begin{aligned} \xi^e = & \frac{\ell}{2} e^{-r/\ell} \left(f^+(e^{2r/\ell} - \mathcal{L}^+) + f^-(e^{2r/\ell} - \mathcal{L}^-) + \frac{1}{2}(f^{+''} + f^{-''}) \right) T^0 \\ & + \frac{\ell}{2} e^{-r/\ell} \left(f^+(e^{2r/\ell} + \mathcal{L}^+) - f^-(e^{2r/\ell} + \mathcal{L}^-) - \frac{1}{2}(f^{+''} - f^{-''}) \right) T^1 \\ & - \frac{1}{2}(f^{+'} + f^{-'}) T^2 \end{aligned} \quad (4.14)$$

and

$$\begin{aligned} \xi^\Omega = & -\frac{\ell}{2} e^{-r/\ell} \left(f^+(e^{2r/\ell} - \mathcal{L}^+) - f^-(e^{2r/\ell} - \mathcal{L}^-) + \frac{1}{2}(f^{+''} - f^{-''}) \right) T^0 \\ & - \frac{\ell}{2} e^{-r/\ell} \left(f^+(e^{2r/\ell} + \mathcal{L}^+) + f^-(e^{2r/\ell} + \mathcal{L}^-) - \frac{1}{2}(f^{+''} + f^{-''}) \right) T^1 \\ & + \frac{1}{2}(f^{+'} - f^{-'}) T^2. \end{aligned} \quad (4.15)$$

These gauge parameters solve (4.2) with state dependent functions \mathcal{L}^\pm transforming as CFT stress tensors

$$\delta_\xi \mathcal{L}^\pm = f^\pm \mathcal{L}^{\pm'} + 2f^{\pm'} \mathcal{L}^\pm - \frac{1}{2} f^{\pm''}. \quad (4.16)$$

The next step is to compute the variation of the charges (4.5) and check whether it is well-defined, finite and integrable. The result we obtain is all of those things and integrates to

$$Q^\pm[f^\pm] = \frac{\ell}{8\pi G} \left(\sigma \pm \frac{1}{\mu\ell} + \alpha C \right) \int d\varphi f^\pm(x^\pm) \mathcal{L}^\pm(x^\pm). \quad (4.17)$$

Finally, using (4.6) together with the transformation properties of the functions \mathcal{L}^\pm (4.16), we find that the asymptotic symmetry algebra is given by two copies of the Virasoro algebra for the Fourier modes of \mathcal{L}^\pm with central charges

$$c^\pm = \frac{3\ell}{2G} \left(\sigma \pm \frac{1}{\mu\ell} + \alpha C \right) = \pm \frac{3\ell}{2G} \frac{\lambda_\mp}{\mu\ell}. \quad (4.18)$$

This result shows that positivity of both c^+ and c^- is equivalent to $\pm\lambda_\mp > 0$, as claimed in subsection 3.3. It also agrees with [2] but differs from the result of [6], which is based on a quadratic action that is inequivalent to the quadratic approximation to the non-linear MMG action.

4.3 BTZ thermodynamics

For constant $\mathcal{L}^\pm = \frac{2G}{\ell}(\ell\mathbf{m} \pm \mathbf{j})$ the Bañados solutions (4.7) describe BTZ black holes in Einstein gravity with mass \mathbf{m} and angular momentum \mathbf{j} . These metrics also solve the MMG field equations, but the mass and angular momentum get extra contributions. Using the results of the last section it is particularly easy to compute the BTZ mass in MMG, which corresponds to the asymptotic charge for time translations. From (4.3) we see that the gauge parameter corresponding to a time translation is simply $\xi^p = a_t^p$. By inspection of (4.8b) and (4.14) one can easily verify that this corresponds to choosing $f^\pm = 1$. The BTZ mass is now readily computed from (4.17) as

$$\ell M_{MMG} = Q^+[f^+ = 1] + Q^-[f^- = 1] = (\sigma + \alpha C) \ell\mathbf{m} + \frac{\mathbf{j}}{\mu\ell}. \quad (4.19)$$

Similarly, the angular momentum of the black hole, corresponding to the asymptotic charge associated to the Killing vector ∂_φ , is easily obtained as:

$$J_{MMG} = Q^+[f^+ = 1] + Q^-[f^- = -1] = (\sigma + \alpha C) \mathbf{j} + \frac{\mathbf{m}}{\mu}. \quad (4.20)$$

The mass and angular momentum satisfy the first law of black hole thermodynamics when the entropy of the BTZ black hole in MMG is given by

$$S = \frac{2\pi}{4G} \left((\sigma + \alpha C)r_+ + \frac{1}{\mu\ell}r_- \right) = \frac{\pi}{6\ell} (c^+(r_+ + r_-) + c^-(r_+ - r_-)), \quad (4.21)$$

where r_\pm are the horizon radii of the BTZ black hole; these are given in terms of \mathbf{m} and \mathbf{j} by

$$r_\pm = \sqrt{2G\ell(\ell\mathbf{m} + \mathbf{j})} \pm \sqrt{2G\ell(\ell\mathbf{m} - \mathbf{j})}. \quad (4.22)$$

The microscopic Cardy formula for the entropy in the canonical ensemble is (see, e.g. [17])

$$S = \frac{\pi^2 \ell}{3} (c^+ T^+ + c^- T^-) , \quad (4.23)$$

where T^\pm are the left and right temperatures. Identifying these as the temperatures of the outer and inner Killing horizons, which are $T^\pm = (r_+ \pm r_-)/(2\pi\ell^2)$, one recovers the macroscopic entropy formula (4.21).

5 Discussion

The massive 3D gravity models TMG and MMG both propagate a single massive spin-2 mode. Although they differ in their interactions, linearization about an AdS_3 background yields locally equivalent field equations. Nevertheless, the quadratic actions of linearized TMG and MMG are inequivalent. This is possible because these quadratic actions include, for an appropriate basis of fields, a 3D linearized Einstein-Hilbert term in addition to an action for the spin-2 mode, and this introduces a relative sign that cannot be changed by field redefinitions. Moreover, this relative sign is physically significant because it determines whether there will be a concordance or a clash between the twin requirements of positive energy for the massive graviton and positive Virasoro central charges for the asymptotic conformal symmetry algebra, both of which are required for semi-classical unitarity. Semi-classical MMG avoids this clash within a region of its two-parameter space, whereas semi-classical TMG does not, and this is possible because the linearized action of MMG is inequivalent to the linearized action of TMG.

In other words, the on-shell equivalence of linearized TMG and NMG does not imply an off-shell equivalence of the quadratic approximations to the TMG and MMG actions. This is important because all semi-classical unitarity conditions are constraints on the coefficients of the terms in this action, and these coefficients, are determined (up to an overall factor) by the respective *inequivalent* interactions.

We have spelled this out in detail here because the distinction between on-shell and off-shell inequivalence of linearized TMG and MMG is a subtle one that has been overlooked in other discussions in the literature on these massive 3D gravity models. However, the distinction is an obvious one in the context of another pair of 3D massive gravity theories, even when linearized about a Minkowski vacuum: this pair is “New Massive Gravity” (NMG) [18] and (the third-way consistent) “Exotic Massive Gravity” (EMG) [19].

The field equations of NMG and EMG become equivalent when linearized about a Minkowski vacuum: they both propagate a parity doublet of spin-2 modes. However, the quadratic action for linearized EMG is inequivalent to that of linearized NMG. This is because the EMG action is parity odd whereas the NMG action is parity even and this distinction survives in the quadratic approximation, even though the linearized field equations are equivalent. A similar result holds for linearization about an AdS

vacuum, with important implications for the two Virasoro charges of the asymptotic conformal algebra: they are equal in magnitude for both NMG and EMG but they have the same sign for NMG and opposite sign for EMG.

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